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Linear dynamics of double-porosity dual-permeability materials. I. Governing equations and acoustic attenuation

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The equations governing the linear acoustics of composites with two isotropic porous constituents are derived from first principles using volume-averaging arguments. The theory is designed for modeling acoustic propagation through heterogeneous porous structures. The only restriction placed on the geometry of the two porous phases is that the overall composite remains isotropic. The theory determines the macroscopic fluid response in each porous phase in addition to the combined bulk response of the grains and fluid in the composite. The complex frequency-dependent macroscopic compressibility laws that are obtained allow for fluid transfer between the porous constituents. Such mesoscopic fluid transport between constituents within each averaging volume provides a distinct attenuation mechanism from the losses associated with the net Darcy flux within individual constituents as is quantified in the examples.

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I. INTRODUCTION

Most natural porous materials such as rocks and sediments in the earth have heterogeneity in the porous-continuum properties at nearly all scales greater than grain scales (i.e., scales $\geqslant 1$ mm). Seismic wavelengths used for oil and gas exploration purposes are typically in the range from 1 to 100 m. Seismic forward modeling is therefore performed by discretizing the earth into elements that have linear dimensions ranging from tens of centimeters to tens of meters and typically only the average response in such elements is measured and modeled. When seismic waves stress an averaging element containing mesoscopic heterogeneity (heterogeneity at scales greater than grain sizes but less than wavelengths), pore fluids in different porosity types respond with different changes in their fluid pressures. An internal equilibration then takes place with fluid flowing from the more compliant high-pressure regions to the relatively stiff low-pressure regions. Such mesoscopic flow attenuates significant amounts of wave energy. For purposes of attenuating unwanted noise, one could, for example, design sound-absorbing materials containing mesoscopic heterogeneity capable of reducing vibrations within a given frequency band.

However, no general theory has yet been developed for wave propagation through materials containing mesoscopic heterogeneity. Our approach here is to make the idealization that the mesoscopic structure can be meaningfully reduced to a mixture of just two porous continua. We make such a ‘‘double-porosity’’ idealization so that analytical results may be obtained for the nature of the differential equations and coefficients controlling the macroscopic response. No restrictions are placed on the mesoscopic geometry of the two porosity constituents other than the composite as a whole is assumed to be isotropic.

This is the first of two papers dedicated to obtaining the macroscopic governing equations for such double-porosity dual-permeability composite materials. In this first paper (Paper I), the governing equations are derived and the sources of acoustic attenuation in such materials are clarified. An essential part of the analysis is to model properly the frequency dependence of the internal mesoscopic flow between the two constituents. Thus, in the second paper (Paper II), the fluid transport laws governing the wave-induced fluid flow will be developed in greater detail. The Biot theory of porous-media acoustics [1,2] ignores all wave-induced flow at mesoscopic scales. It is well known that Biot’s theory is not capable of simultaneously explaining both the velocity and attenuation data measured on porous rocks [3]. The theory presented in these two papers provides one method for doing so.

Double-porosity models were originally developed for modeling the fluid flow during pumping of earth reservoirs. Early models of Barenblatt and Zheltov [4] and Warren and Root [5] assumed the pumping did not affect the state of the geological material; however, for the continued pumping of interest to the oil and gas industry, it is necessary to account for the pore-space reduction. Thus, more sophisticated models were developed [6–28] allowing for the coupling between rock deformation and fluid flow. At very slow rates of applied-stress variation, the fluid pressure has enough time to equilibrate internally between the two porous phases and such double-porosity models must then reduce to the usual single-porosity mechanics established by Biot [1,2].

In addition to such double-porosity modeling, there has been considerable work focusing on flow between the layers of a plane-stratified material due to compressional waves propagating normal to the layering [29–32]. The present study seeks to model the flow for arbitrary mesoscopic geometry, albeit under the restriction to only two porous phases

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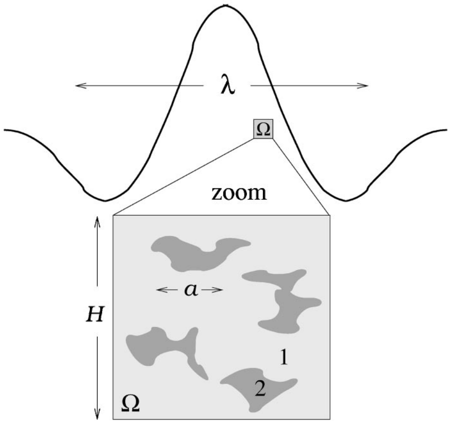

FIG. 1. (Color online) The length scale relations $\lambda > H > a$ where λ is the wavelength of the acoustic pulse, H is the linear dimension of the averaging volume Ω being employed, and a is a characteristic dimension of the mesoscopic-scale heterogeneity. Both phases 1 and 2 are taken to be isotropic porous continua.

mixed together in each averaging volume.

In Sec. II, the porous-continuum laws that control the local response of each isotropic porous material are stated. In Sec. III, the general form of the macroscopic constitutive equations are obtained from the macroscopic statement of the energy balance. The local laws are then integrated over an averaging element of the composite in the sections that follow. In Sec. IV, the macroscopic statement of the conservation of linear momentum is obtained while in Sec. V, the detailed nature of the coefficients in the macroscopic compressibility laws are obtained and in Sec. VII, the macroscopic shear constitutive law is discussed. In Sec. VIII, we summarize the entire set of macroscopic laws, and in Sec. IX reduce the double-porosity theory to an effective single-porosity Biot theory. Finally, we give examples of the P-wave attenuation and then summarize our conclusions. Paper II [33] analyzes the fluid transport laws governing the wave-induced fluid flow.

II. GOVERNING EQUATIONS FOR THE LOCAL RESPONSE

Within each macroscopic averaging volume Ω, we assume there is a welded composite (i.e., material points initially in contact remain in contact) of two isotropic porous materials. To produce useful results, such averaging regions must have linear dimensions H smaller than the wavelengths λ of the applied stress field but larger than a characteristic dimension a of the phases within the composite. These length relations are schematically depicted in Fig. 1. In the present study, a single fluid is assumed to saturate both porous phases, but with minor modifications the same formalism applies to partial and patchy saturation as well. We now define the boundary-value problem controlling such local response in Ω.

Using an index i to denote the two phases ($i=1$ or 2), the differential equations controlling the local response in each material are taken to be Biot's equations [1,2], written in the form

$$\begin{bmatrix} \nabla\cdot\dot{\mathbf{u}}_i \\ \nabla\cdot\mathbf{Q}_i \end{bmatrix} = -\frac{1}{K_i}\begin{bmatrix} 1 & -\alpha_i \\ -\alpha_i & \alpha_i/B_i \end{bmatrix}\begin{bmatrix} \dot{p}_{ci} \\ \dot{p}_{fi} \end{bmatrix}, \tag{1}$$

$$\mathbf{Q}_i = -\frac{k_i}{\eta}(\nabla p_{fi} + \rho_f \ddot{\mathbf{u}}_i - \rho_f \mathbf{g}), \tag{2}$$

$$\nabla\cdot\boldsymbol{\tau}_i^D - \nabla p_{ci} = \rho\ddot{\mathbf{u}}_i + \rho_f\dot{\mathbf{Q}}_i - \rho\mathbf{g}, \tag{3}$$

$$\boldsymbol{\tau}_i^D = G_i\left(\nabla\mathbf{u}_i + \nabla\mathbf{u}_i^T - \frac{2}{3}\nabla\cdot\mathbf{u}_i\mathbf{I}\right). \tag{4}$$

The response fields here are themselves volume averages taken over a scale larger than the grain sizes but smaller than the mesoscopic extent a of either phase. The response fields are the following: $\mathbf{u}_i$, the average displacement of the framework of grains; $\mathbf{Q}_i$, the local Darcy filtration velocity; p_{fi}, the fluid pressure; p_{ci}, the confining pressure acting on the porous continuum (total average pressure); and $\boldsymbol{\tau}_i^D$, the deviatoric (or shear) stress tensor acting on the porous continuum. In the linear theory being developed here, the overdots on these fields denote a partial time derivative. The acceleration $\mathbf{g}$ represents gravity in the case of a consolidation problem; however, in the present context of waves, it is taken as the proxy for a concentrated seismic source. In the Darcy law [Eq. (2)], η is the fluid viscosity and the permeability k_i is a linear time-convolution operator defined so that a term such as $k_i\nabla p_{fi}$ implies

$$k_i\nabla p_{fi} \equiv \frac{1}{2\pi}\int_0^\infty dt' \int_{-\infty}^\infty d\omega e^{-i\omega t'} k_i(\omega)\nabla p_{fi}(\mathbf{r}, t-t'), \tag{5}$$

where the complex "dynamic" permeability of each phase $k_i(\omega)$ has been explicitly and properly modeled in Ref. [34] to allow for the development of viscous boundary layers in the pores of phase i. In the bulk-force balance [Eq. (3)], ρ is the bulk density (volume average of solid and fluid within Ω), while ρ_f is just the fluid density.

In Eq. (1), K_i is the drained bulk modulus of phase i, B_i is Skempton's coefficient [35] of phase i (fluid-pressure change divided by confining-pressure change for a sealed sample), and α_i is the Biot-Willis coefficient [36] of phase i defined as

$$\alpha_i = (1 - K_i/K_i^u)/B_i, \tag{6}$$

where K_i^u is Gassmann's [37] undrained bulk modulus (confining-pressure change divided by sample dilatation for a sealed sample). In the present work, no restrictions to single-mineral isotropic grains will be made. Finally, in the deviatoric constitutive law [Eq. (4)], G_i is the shear modulus of the framework of grains. All of these material properties are taken to be uniform throughout their respective phases.

We finally state the boundary conditions to be specified on the surface $\partial\Omega$ surrounding the averaging volume. Either of $\mathbf{n}\cdot\boldsymbol{\tau}$ or $\dot{\mathbf{u}}$ and either of p_f or $\mathbf{n}\cdot\mathbf{Q}$ must be specified at each point of $\partial\Omega$ if a unique local response is to be obtained [38–40]. Here, $\mathbf{n}$ is the outward normal to the averaging-volume external surface. The specific values we adopt for these boundary fields are presented in the following section.

Finally, the continuity conditions (at the welded contacts) on the internal interface $\partial\Omega_{12}$ separating the two phases within each averaging volume are $\mathbf{n}\cdot\boldsymbol{\tau}_1=\mathbf{n}\cdot\boldsymbol{\tau}_2$, $p_{f1}=p_{f2}$, $\dot{\mathbf{u}}_1=\dot{\mathbf{u}}_2$, and $\mathbf{n}\cdot\mathbf{q}_1=\mathbf{n}\cdot\mathbf{q}_2$. Such continuity conditions are what couple the averaged response in phase 1 to the averaged response in phase 2.

III. MACROSCOPIC FORM OF THE CONSTITUTIVE EQUATIONS

To determine the macroscopic form that the constitutive equations must take, we focus here on the rate $\dot{E}$ at which the energy in an averaging volume is changing (as normalized by the volume V of the averaging region). Such changes in the energy density are entirely due to the rate at which a wave is doing work on an averaging volume and are thus given by

$$\dot{E}=\frac{1}{V}\sum_{i=1}^{2}\int_{\partial\Omega_i}[\mathbf{n}\cdot\boldsymbol{\tau}_i\cdot\dot{\mathbf{u}}_i-\mathbf{n}\cdot\mathbf{Q}_i p_{fi}]dS, \tag{7}$$

where the exterior surface of the averaging volume is being partitioned into phase 1 and phase 2 portions as $\partial\Omega=\partial\Omega_1+\partial\Omega_2$. In this section, we first express $\dot{E}$ in terms of macroscopic field variables and then use the result to write down the form of the constitutive equations.

The boundary values on $\partial\Omega$ are the source terms for the local response within Ω and are directly related to the macroscopic forcing provided by a passing wave. The local particle velocities $\dot{\mathbf{u}}_i$ have boundary values that may be written as

$$\dot{\mathbf{u}}_i(\mathbf{x})=\mathbf{v}+\mathbf{x}\cdot\boldsymbol{\nabla}\mathbf{v}+\delta\dot{\mathbf{u}}_i(\mathbf{x}) \quad \text{on} \quad \partial\Omega_i, \tag{8}$$

where $\mathbf{x}=\mathbf{r}'-\mathbf{r}$ represents distance to $\partial\Omega$ as measured from the center point $\mathbf{r}$ of each averaging volume, and where $\mathbf{v}$ is the average velocity of the solid phase throughout the entire averaging volume,

$$\mathbf{v}=\frac{1}{V}\left[\int_{\Omega_1}\dot{\mathbf{u}}_1 dV+\int_{\Omega_2}\dot{\mathbf{u}}_2 dV\right] \tag{9}$$

and is thus a constant vector on $\partial\Omega$. The gradient of the average is given in this case by (cf. Ref. [40])

$$\boldsymbol{\nabla}\mathbf{v}=\frac{1}{V}\left[\int_{\partial\Omega_1}\mathbf{n}\dot{\mathbf{u}}_1 dS+\int_{\partial\Omega_2}\mathbf{n}\dot{\mathbf{u}}_2 dS\right]. \tag{10}$$

The first term of Eq. (8) represents a rigid-body translation induced by the wave, the second term represents the smooth wave-induced deformation, while the third term $\delta\dot{\mathbf{u}}_i$ represents any left over higher-order modes of "bumpiness" that develop due to local heterogeneity. By substituting the boundary values of Eq. (8) into Eq. (10), it can be verified that the perturbations $\delta\dot{\mathbf{u}}_i$ due to heterogeneity do not contribute to the macroscopic deformation rate $\boldsymbol{\nabla}\mathbf{v}$ (i.e., $\int_{\partial\Omega_i}\mathbf{n}\delta\dot{\mathbf{u}}_i dS=0$).

Formally, one can express the leftover $\delta\dot{\mathbf{u}}_i$ using higher-order modes (tensorial orders) of the macroscopic response; e.g., the next boundary term is $\delta\dot{\mathbf{u}}_i(\mathbf{x})=\mathbf{x}\mathbf{x}:\boldsymbol{\nabla}\boldsymbol{\nabla}\mathbf{v}/2$ on $\partial\Omega_i$, while additional terms would involve still higher-order gradients of the macroscopic response. Such terms are important only if there is simultaneously present an equivalent spatial distribution in stress so that non-negligible contributions to Eq. (7) are made. For example, in a laboratory experiment, where a uniform stress tensor is applied to the surface of a heterogeneous sample, the heterogeneity will necessarily produce the higher-order terms $\delta\dot{\mathbf{u}}_i(\mathbf{x})$ in addition to the smooth response $\mathbf{x}\cdot\boldsymbol{\nabla}\mathbf{v}$; however, there will be no net strain energy stored in such higher-order deformation, as Eq. (7) makes clear.

Obtaining precise conditions for the neglect of these higher-order work terms is rather involved, since they depend on details of the long range correlation of the heterogeneity (over scales necessarily larger than the size H of the averaging volume). Retaining such higher-order terms causes gradients of strain (i.e., third- and higher-order tensors) to be explicitly present in the macroscopic laws along with their associated stress-moment tensors (also third- and higher-order tensors). However, if analysis is limited to macroscopically isotropic double-porosity composites, such higher-order response exactly decouples from the smooth homogenized response of interest because these responses are of different tensorial order (a result known as "Pierre Curie's principle" [41]). The analysis of the present theory is thus restricted to such macroscopically isotropic composites.

For the fluid boundary conditions, the dual-permeability nature of the materials being studied requires us to consider separately (and independently) the average fluid response in both phases 1 and 2,

$$\overline{p}_{fi}=\frac{1}{V_i}\int_{\Omega_i}p_{fi}dV, \tag{11}$$

where V_i is the volume of phase i in the averaging volume. As a wave compresses each averaging volume, it changes $\overline{p}_{fi}$ in each phase while simultaneously producing wavelength scale (macroscopic) gradients in the averaged fluid pressure $\boldsymbol{\nabla}\overline{p}_{fi}$. These gradients along with the apparent macroscopic force terms $\rho_f(\mathbf{g}-\dot{\mathbf{v}})$ produce a Darcy flow in each phase. Because such macroscopic forces themselves vary over a wavelength, fluid accumulates (or depletes) in each phase of each averaging volume. Allowing for these fluid accumulations is a key part of the present analysis.

The boundary conditions for such flows are the fluid-pressure conditions

$$p_{fi}(\mathbf{x})=\overline{p}_{fi}+\mathbf{x}\cdot\mathbf{f}_i+\delta p_{fi}(\mathbf{x}) \quad \text{on} \quad \partial\Omega_i, \tag{12}$$

where the vector $\mathbf{f}_i$ represents the macroscopic forces driving relative fluid-solid flow in each phase and is given by

$$\mathbf{f}_i = -\nabla \overline{p}_f + \rho_f(\mathbf{g} - \dot{\mathbf{v}}). \tag{13}$$

The perturbations $\delta p_{fi}(\mathbf{x})$ are again due to the local heterogeneity and will again be neglected in Eq. (7) [i.e., contributions such as $\delta p_{fi}(\mathbf{x}) = \mathbf{xx}:\nabla \mathbf{f}_i$, and so on, correspond to higher-order tensorial orders of the response that are decoupled from the smooth response being allowed for when the composite is isotropic].

Thus, the closure conditions of our theory (i.e., those conditions that permit a closed set of macroscopic equations to be obtained which are capable of uniquely determining a finite number of macroscopic response fields) are the boundary conditions

$$\delta\dot{\mathbf{u}}_i(\mathbf{x}) = 0 \quad \text{and} \quad \delta p_{fi}(\mathbf{x}) = 0 \quad \text{on} \quad \partial\Omega_i. \tag{14}$$

Such perturbations from the smoothed response are only taken to be zero on the boundary of each averaging volume. They are necessarily nonzero throughout the interior of Ω in order to satisfy the local Biot equations there. As stated above, these closure conditions lead to the proper governing equations for the smoothed response whenever the material is macroscopically isotropic.

The boundary conditions of Eqs. (8), (12), and (14) are now inserted into the surface integrals of Eq. (7) to obtain

$$V\dot{E} = \sum_{i=1}^{2} \left[\mathbf{v} \cdot \int_{\partial\Omega_i} \mathbf{n} \cdot \boldsymbol{\tau}_i dS + \dot{\mathbf{e}} : \int_{\partial\Omega_i} \mathbf{n} \cdot \boldsymbol{\tau}_i \mathbf{x} dS - \overline{p}_{fi} \int_{\partial\Omega_i} \mathbf{n} \cdot \mathbf{Q}_i dS \right.$$
$$\left. - \nabla \overline{p}_{fi} \cdot \int_{\partial\Omega_i} \mathbf{n} \cdot \mathbf{Q}_i \mathbf{x} dS \right], \tag{15}$$

where $\dot{\mathbf{e}} = [\nabla \mathbf{v} + (\nabla \mathbf{v})^T]/2$ is the macroscopic strain-rate tensor. The four integrals of the local fields here are now identified as macroscopic field variables.

For an arbitrary local field ψ_i in phase i, the definition of the gradient of a volume-averaged field is (e.g., Ref. [40])

$$\nabla(v_i \overline{\psi_i}) = \frac{1}{V} \int_{\partial\Omega_i} \mathbf{n} \psi_i dS, \tag{16}$$

where $v_i = V_i/V$ is the volume fraction of phase i present in an averaging volume. Thus, we find immediately that

$$\nabla \cdot (v_i \overline{\boldsymbol{\tau}_i}) = \frac{1}{V} \int_{\partial\Omega_i} \mathbf{n} \cdot \boldsymbol{\tau}_i dS. \tag{17}$$

The identity $\nabla \cdot (\boldsymbol{\tau}_i \mathbf{x}) = (\nabla \cdot \boldsymbol{\tau}_i)\mathbf{x} + \boldsymbol{\tau}_i^T$ is now integrated over Ω_i and the divergence theorem applied to obtain

$$\sum_{i=1}^{2} \frac{1}{V} \int_{\partial\Omega_i} \mathbf{n} \cdot \boldsymbol{\tau}_i \mathbf{x} dS = \overline{\boldsymbol{\tau}} + \sum_{i=1}^{2} \frac{1}{V} \int_{\Omega_i} (\nabla \cdot \boldsymbol{\tau}_i) \mathbf{x} dV \tag{18}$$

$$= \overline{\boldsymbol{\tau}}[1 + O(H/\lambda)]. \tag{19}$$

Here, the volume-average expression

$$\overline{\boldsymbol{\tau}} = v_1 \overline{\boldsymbol{\tau}_1} + v_2 \overline{\boldsymbol{\tau}_2} \tag{20}$$

represents the average total stess tensor in each averaging volume, while H is a characteristic dimension of the averaging volume, and λ is the macroscopic wavelength. A fundamental condition for the averaging is that $H/\lambda \ll 1$. That the volume integral in Eq. (18) is H/λ times smaller than the average stress tensor (and, therefore, negligible) follows because the only spatial variations of $(\nabla \cdot \boldsymbol{\tau}_i)\mathbf{x}$ that do not integrate to zero over the volume are the wavelength-scale variations. The surface integrals over the internal surface $\partial\Omega_{12}$ that arise when the divergence theorem is applied have exactly vanished from Eq. (18) due to stress continuity on $\partial\Omega_{12}$ once the sum over i is performed (note that the total closed surface around each phase is $\partial\Omega_i + \partial\Omega_{12}$).

Again from Eq. (16), we have

$$\nabla \cdot (v_i \overline{\mathbf{Q}_i}) = \frac{1}{V} \int_{\partial\Omega_i} \mathbf{n} \cdot \mathbf{Q}_i dS, \tag{21}$$

while we introduce the definition of macroscopic fluid flux $\mathbf{q}_i$ through each phase as

$$\mathbf{q}_i = \frac{1}{V} \int_{\partial\Omega_i} \mathbf{n} \cdot \mathbf{Q}_i \mathbf{x} dS. \tag{22}$$

We now demonstrate that

$$\lim_{H/\lambda \to 0} \nabla \cdot \mathbf{q}_i = \nabla \cdot (v_i \overline{\mathbf{Q}_i}). \tag{23}$$

To do so, consider the definition of the directional derivative of $\mathbf{q}_i$,

$$d\mathbf{s} \cdot \nabla \mathbf{q}_i(\mathbf{r}) = \mathbf{q}_i(\mathbf{r} + d\mathbf{s}) - \mathbf{q}_i(\mathbf{r}), \tag{24}$$

valid as $|d\mathbf{s}| \to 0$. A normal projection of the integrand of Eq. (22) from the surface $\partial\Omega_i(\mathbf{r} + \mathrm{d}\mathbf{s})$ to the surface $\partial\Omega_i(\mathbf{r})$ yields

$$d\mathbf{s} \cdot \nabla \mathbf{q}_i(\mathbf{r}) = \frac{1}{V} \int_{\partial\Omega_i(\mathbf{r})} d\mathbf{s} \cdot \mathbf{n}(\mathbf{n}\, \mathbf{Q}_i \cdot \mathbf{n} - \mathbf{n} \cdot \nabla \mathbf{Q}_i \cdot \mathbf{n}\, \mathbf{x}) dS$$
$$+ O(|d\mathbf{s}|^2). \tag{25}$$

Because the second term of the integrand is again $O(H/\lambda)$ relative to the first (and, therefore, negligible), we find that since the direction of d$\mathbf{s}$ is arbitrary,

$$\nabla \mathbf{q}_i = \frac{1}{V} \int_{\partial\Omega_i} \mathbf{nn} \mathbf{Q}_i \cdot \mathbf{n} dS [1 + O(H/\lambda)]. \tag{26}$$

Taking the trace of this equation gives exactly Eq. (21), thus proving that $\nabla \cdot \mathbf{q}_i = \nabla \cdot (v_i \overline{\mathbf{Q}_i})$ whenever $H \ll \lambda$.

Last, we define the rate $\dot{\zeta}_{\text{int}}$ at which fluid volume is flowing from phase 1 into phase 2 (i.e., ζ_{int} is the increment in fluid content for fluid in phase 2 due to internal diffusion across $\partial\Omega_{12}$),

$$\dot{\zeta}_{\text{int}} = \frac{1}{V} \int_{\partial\Omega_{12}} \mathbf{n} \cdot \mathbf{Q}_1 dS, \tag{27}$$

where the normal $\mathbf{n}$ is directed from phase 1 toward phase 2. Thus, $\nabla\cdot\mathbf{q}_1+\dot{\zeta}_{\mathrm{int}}$ is the total rate at which fluid is being depleted from phase 1, while $\nabla\cdot\mathbf{q}_2-\dot{\zeta}_{\mathrm{int}}$ is the rate at which fluid is depleting from phase 2.

With all these definitions in hand, we finally can rewrite Eq. (15) in the instructive form

$$\dot{E}=\mathbf{v}\cdot(\nabla\cdot\overline{\boldsymbol{\tau}})+\tfrac{1}{2}[\nabla\mathbf{v}+(\nabla\mathbf{v})^T]:\overline{\boldsymbol{\tau}}-(\nabla\cdot\mathbf{q}_1+\dot{\zeta}_{\mathrm{int}})\overline{p}_{f1}-(\nabla\cdot\mathbf{q}_2-\dot{\zeta}_{\mathrm{int}})\overline{p}_{f2}+\dot{\zeta}_{\mathrm{int}}(\overline{p}_{f1}-\overline{p}_{f2})+\mathbf{q}_1\cdot\mathbf{f}_1+\mathbf{q}_2\cdot\mathbf{f}_2. \tag{28}$$

To exploit this result, the various contributions are grouped and identified.

The first term $\mathbf{v}\cdot(\nabla\cdot\overline{\boldsymbol{\tau}})$ is the rate at which kinetic energy is changing, since the conservation of linear momentum requires $\nabla\cdot\overline{\boldsymbol{\tau}}$ to be equal to the total inertial acceleration (as demonstrated in the following section).

To consider the next terms, we first write $\overline{\boldsymbol{\tau}}$ in terms of isotropic and deviatoric contributions:

$$\overline{\boldsymbol{\tau}}=-\overline{p}_c\mathbf{I}+\overline{\boldsymbol{\tau}}^D, \tag{29}$$

where $\overline{p}_c=-\mathrm{tr}\{\overline{\boldsymbol{\tau}}\}/3$ is the total confining pressure acting on Ω. We further separate the deviatoric stress into elastic and viscous contributions

$$\overline{\boldsymbol{\tau}}^D=\overline{\boldsymbol{\tau}}^D_e+\overline{\boldsymbol{\tau}}^D_v. \tag{30}$$

As is discussed in Sec. VII, the sudden application of a shear deformation can result in localized pressure gradients within Ω that then equilibrate. Such a transient viscoelastic effect is represented by $\overline{\boldsymbol{\tau}}^D_v$. The strain-rate tensor is also written in terms of isotropic and deviatoric contributions as

$$\dot{\mathbf{e}}=\tfrac{1}{3}\nabla\cdot\mathbf{v}\,\mathbf{I}+\dot{\mathbf{e}}^D, \tag{31}$$

$$\dot{\mathbf{e}}^D=\frac{1}{2}\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^T-\frac{2}{3}\nabla\cdot\mathbf{v}\mathbf{I}\right]. \tag{32}$$

With these identifications, the rate $\dot{R}$ at which energy is being reversibly stored as strain energy may be written as

$$\dot{R}=\overline{\boldsymbol{\tau}}^D_e:\dot{\mathbf{e}}^D-\overline{p}_c\nabla\cdot\mathbf{v}-\overline{p}_{f1}(\nabla\cdot\mathbf{q}_1+\dot{\zeta}_{\mathrm{int}})-\overline{p}_{f2}(\nabla\cdot\mathbf{q}_2-\dot{\zeta}_{\mathrm{int}}). \tag{33}$$

This statement allows us to deduce that R has the functional dependence

$$R=R(\mathbf{e}^D,\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{w}_1+\zeta_{\mathrm{int}},\nabla\cdot\mathbf{w}_2-\zeta_{\mathrm{int}}), \tag{34}$$

where the displacements $\mathbf{u}$ and $\mathbf{w}_i$ are defined from $\mathbf{v}=\dot{\mathbf{u}}$ and $\mathbf{q}_i=\dot{\mathbf{w}}_i$. Upon taking the total derivative of this R and comparing to the derived statement of Eq. (33), we have

$$\overline{\boldsymbol{\tau}}^D_e=\frac{\partial R}{\partial\mathbf{e}^D}, \tag{35}$$

$$-\overline{p}_c=\frac{\partial R}{\partial(\nabla\cdot\mathbf{u})}, \tag{36}$$

$$-\overline{p}_{f1}=\frac{\partial R}{\partial(\nabla\cdot\mathbf{w}_1+\zeta_{\mathrm{int}})}, \tag{37}$$

$$-\overline{p}_{f2}=\frac{\partial R}{\partial(\nabla\cdot\mathbf{w}_2-\zeta_{\mathrm{int}})}, \tag{38}$$

which are thus the state functions of the theory.

Since our treatment is limited to isotropic macroscopic laws, the various tensorial orders of the response are decoupled [41]. Thus, upon taking the time derivative of Eqs. (35)–(38), we obtain isotropic stress-strain relations for the double-porosity composite in the form

$$\dot{\overline{\boldsymbol{\tau}}}^D_e=G\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^T-\frac{2}{3}\nabla\cdot\mathbf{v}\mathbf{I}\right], \tag{39}$$

$$\begin{bmatrix}\dot{\overline{p}}_c\\ \dot{\overline{p}}_{f1}\\ \dot{\overline{p}}_{f2}\end{bmatrix}=-\begin{bmatrix}c_{11}&c_{12}&c_{13}\\ c_{12}&c_{22}&c_{23}\\ c_{13}&c_{23}&c_{33}\end{bmatrix}\begin{bmatrix}\nabla\cdot\mathbf{v}\\ \nabla\cdot\mathbf{q}_1+\dot{\zeta}_{\mathrm{int}}\\ \nabla\cdot\mathbf{q}_2-\dot{\zeta}_{\mathrm{int}}\end{bmatrix}. \tag{40}$$

The coefficients in these laws are the second partial derivatives of R with respect to the appropriate strain. For this reason, we also have the Maxwell relations $c_{ij}=c_{ji}$. Detailed expressions for these coefficients in terms of the underlying material properties are obtained in Sec. V.

We next denote

$$\sigma=\dot{\zeta}_{\mathrm{int}}(\overline{p}_{f1}-\overline{p}_{f2})+\mathbf{q}_1\cdot\mathbf{f}_1+\mathbf{q}_2\cdot\mathbf{f}_2+\overline{\boldsymbol{\tau}}^D_v:\dot{\mathbf{e}}^D \tag{41}$$

as the rate at which energy is being dissipated ($\sigma>0$ and is not the derivative of a thermodynamic potential). Upon appealing to isotropic macroscopic media and the standard argument of irreversible thermodynamics, we expect linear transport laws of the form

$$\dot{\zeta}_{\mathrm{int}}=\Gamma(\overline{p}_{f1}-\overline{p}_{f2}), \tag{42}$$

$$\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\end{bmatrix}=-\frac{1}{\eta}\begin{bmatrix}\kappa_{11}&\kappa_{12}\\ \kappa_{12}&\kappa_{22}\end{bmatrix}\begin{bmatrix}\nabla\overline{p}_{f1}-\rho_f(\mathbf{g}-\dot{\mathbf{v}})\\ \nabla\overline{p}_{f2}-\rho_f(\mathbf{g}-\dot{\mathbf{v}})\end{bmatrix}, \tag{43}$$

$$\overline{\boldsymbol{\tau}}^D_v=F\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^T-\frac{2}{3}\nabla\cdot\mathbf{v}\mathbf{I}\right]. \tag{44}$$

All these transport coefficients can be understood as time-convolution operators. The transport laws and coefficients will be independently derived and analyzed in Paper II.

The above constitutive equations along with the macroscopic statement of the conservation of linear momentum to be derived in the following section provide a closed set of equations for the determination of $\mathbf{v}$, $\mathbf{q}_1$, and $\mathbf{q}_2$.

IV. THE TOTAL-FORCE BALANCE

The macroscopic statement of the conservation of linear momentum is obtained by directly integrating Eq. (3) over the averaging region Ω. By combining Green's theorem for $\overline{\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_i}$ along with Eq. (16) for the gradient of an average, we obtain the so-called averaging theorem [40,42]

$$\frac{1}{V}\int_{\Omega}\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_i dV=\boldsymbol{\nabla}\cdot(v_i\overline{\boldsymbol{\tau}_i})+\frac{1}{V}\int_{\partial\Omega_{12}}\mathbf{n}\cdot\boldsymbol{\tau}_i dS,$$

which allows the total conservation of linear momentum to be expressed as

$$\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\tau}}^{D}-\boldsymbol{\nabla}\overline{p_c}=\rho_1 v_1\ddot{\overline{\mathbf{u}}}_1+\rho_2 v_2\ddot{\overline{\mathbf{u}}}_2+\rho_f(v_1\dot{\overline{\mathbf{Q}}}_1+v_2\dot{\overline{\mathbf{Q}}}_2)$$
$$-(\rho_1 v_1+\rho_2 v_2)\mathbf{g},$$

where we have defined the volume-averaged total stress variables

$$\overline{\boldsymbol{\tau}}^{D}=v_1\overline{\boldsymbol{\tau}}_1^{D}+v_2\overline{\boldsymbol{\tau}}_2^{D}, \tag{45}$$

$$\overline{p}_c=v_1\overline{p}_{c1}+v_2\overline{p}_{c2}. \tag{46}$$

The terms involving the integral of stress over the internal surface $\partial\Omega_{12}$ have vanished due to the continuity of stress. From the fact that flow is continuous across the interface $\partial\Omega_{12}$, we have that $v_1\dot{\overline{\mathbf{Q}}}_1+v_2\dot{\overline{\mathbf{Q}}}_2=\mathbf{q}_1+\mathbf{q}_2$ (even if $v_i\dot{\overline{\mathbf{Q}}}_i\neq\mathbf{q}_i$). Furthermore, if

$$\rho=v_1\rho_1+v_2\rho_2 \tag{47}$$

is defined as the average density of the composite, then the mass-averaged acceleration $\dot{\mathbf{v}}_{\text{mass}}$ is defined by

$$\rho\dot{\mathbf{v}}_{\text{mass}}=\rho_1 v_1\ddot{\overline{\mathbf{u}}}_1+\rho_2 v_2\ddot{\overline{\mathbf{u}}}_2.$$

The difference between the mass-averaged acceleration and the volume-averaged acceleration $\dot{\mathbf{v}}$ is negligible in the linear theory being developed. Thus, we obtain

$$\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\tau}}^{D}-\boldsymbol{\nabla}\overline{p_c}=\rho\dot{\mathbf{v}}+\rho_f\dot{\mathbf{q}}_1+\rho_f\dot{\mathbf{q}}_2-\rho\mathbf{g} \tag{48}$$

as our final statement of the conservation of linear momentum in the composite.

V. MACROSCOPIC COMPRESSIBILITY LAWS

We now obtain detailed expressions for the coefficients appearing in the macroscopic compressibility laws [Eq. (40)]. Upon averaging Eq. (1) over all of Ω, we obtain

$$v_i\begin{bmatrix}\overline{\boldsymbol{\nabla}\cdot\dot{\mathbf{u}}_i}\\ \overline{\boldsymbol{\nabla}\cdot\mathbf{Q}_i}\end{bmatrix}=-\frac{v_i}{K_i}\begin{bmatrix}1 & -\alpha_i\\ -\alpha_i & \alpha_i/B_i\end{bmatrix}\begin{bmatrix}\dot{\overline{p}}_{ci}\\ \dot{\overline{p}}_{fi}\end{bmatrix}, \tag{49}$$

where overbars again denote the average of a field throughout the phase in which it is defined. From the averaging theorem, we have

$$v_1\overline{\boldsymbol{\nabla}\cdot\dot{\mathbf{u}}_1}=\boldsymbol{\nabla}\cdot(v_1\dot{\overline{\mathbf{u}}}_1)+\frac{1}{V}\int_{\partial\Omega_{12}}\mathbf{n}\cdot\dot{\mathbf{u}}_1 dS, \tag{50}$$

$$v_2\overline{\boldsymbol{\nabla}\cdot\dot{\mathbf{u}}_2}=\boldsymbol{\nabla}\cdot(v_2\dot{\overline{\mathbf{u}}}_2)-\frac{1}{V}\int_{\partial\Omega_{12}}\mathbf{n}\cdot\dot{\mathbf{u}}_1 dS, \tag{51}$$

where continuity of $\dot{\mathbf{u}}_i$ on $\partial\Omega_{12}$ has been used. Upon adding Eqs. (50) and (51) and using the definition $\mathbf{v}\equiv v_1\dot{\overline{\mathbf{u}}}_1+v_2\dot{\overline{\mathbf{u}}}_2$, we obtain the exact statement

$$\boldsymbol{\nabla}\cdot\mathbf{v}=-\frac{v_1}{K_1}\dot{\overline{p}}_{c1}-\frac{v_2}{K_2}\dot{\overline{p}}_{c2}+\frac{v_1\alpha_1}{K_1}\dot{\overline{p}}_{f1}+\frac{v_2\alpha_2}{K_2}\dot{\overline{p}}_{f2} \tag{52}$$

for the total rate of change of dilatation for the double-porosity composite.

Next, similar averages are taken over the Darcy-flow accumulations to obtain

$$\boldsymbol{\nabla}\cdot(v_1\overline{\mathbf{Q}}_1)=\frac{v_1\alpha_1}{K_1}\dot{\overline{p}}_{c1}-\frac{v_1\alpha_1}{K_1B_1}\dot{\overline{p}}_{f1}-\dot{\zeta}_{\text{int}}, \tag{53}$$

$$\boldsymbol{\nabla}\cdot(v_2\overline{\mathbf{Q}}_2)=\frac{v_2\alpha_2}{K_2}\dot{\overline{p}}_{c2}-\frac{v_2\alpha_2}{K_2B_2}\dot{\overline{p}}_{f2}+\dot{\zeta}_{\text{int}}. \tag{54}$$

As seen previously, we can justify making the identification $\boldsymbol{\nabla}\cdot\mathbf{q}_i=\boldsymbol{\nabla}\cdot(v_i\overline{\mathbf{Q}}_i)$. In all of Eqs. (52)–(54), the dependence of $\dot{\overline{p}}_{c1}$ and $\dot{\overline{p}}_{c2}$ on the macroscopic deformation variables must be obtained.

Upon making a formal Legendre transformation, a free energy may be obtained from R that has as independent variables not the macroscopic dilatations but rather the pressure variables $\overline{p}_c$, $\overline{p}_{f1}$, and $\overline{p}_{f2}$. Although the average confining pressures $\overline{p}_{ci}$ cannot be written as partial derivatives of the free energy with respect to $\overline{p}_c$, $\overline{p}_{f1}$, and $\overline{p}_{f2}$, they nevertheless must depend on these independent variables. Thus, upon taking the time derivative of $\overline{p}_{ci}=\overline{p}_{ci}(\overline{p}_c,\overline{p}_{f1},\overline{p}_{f2})$, we obtain linear response laws of the form

$$\dot{\overline{p}}_{c1}=Q_1\dot{\overline{p}}_c+S_{11}\dot{\overline{p}}_{f1}+S_{12}\dot{\overline{p}}_{f2}, \tag{55}$$

$$\dot{\overline{p}}_{c2}=Q_2\dot{\overline{p}}_c+S_{21}\dot{\overline{p}}_{f1}+S_{22}\dot{\overline{p}}_{f2}, \tag{56}$$

where the six constants Q_i and S_{ij} must be found. Exact expressions were obtained recently [28] for these constants, so only a sketch of the derivation is given here.

The demonstration amounts to inserting Eqs. (55) and (56) into Eqs. (52)–(54) to obtain linear response laws of the form found earlier [cf. Eq. (40)],

$$\begin{bmatrix}\boldsymbol{\nabla}\cdot\mathbf{v}\\ \boldsymbol{\nabla}\cdot\mathbf{q}_1\\ \boldsymbol{\nabla}\cdot\mathbf{q}_2\end{bmatrix}=-\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}\dot{\overline{p}}_c\\ \dot{\overline{p}}_{f1}\\ \dot{\overline{p}}_{f2}\end{bmatrix}+\begin{bmatrix}0\\ -\dot{\zeta}_{\text{int}}\\ \dot{\zeta}_{\text{int}}\end{bmatrix}. \tag{57}$$

The Maxwell relations $a_{ij}=a_{ji}$ provide three equations for determining the Q_i and S_{ij}. Upon inserting Eqs. (55) and (56) into $\dot{\overline{p}}_c=v_1\dot{\overline{p}}_{c1}+v_2\dot{\overline{p}}_{c2}$, one obtains $0=(v_1Q_1+v_2Q_2-1)\dot{\overline{p}}_c+(v_1S_{11}+v_2S_{21})\dot{\overline{p}}_{f1}+(v_1S_{12}+v_2S_{22})\dot{\overline{p}}_{f2}$, and since each coefficient must vanish, three more equations are obtained. One then finds that

$$v_1Q_1=\frac{1-K_2/K}{1-K_2/K_1}, \tag{58}$$

$$v_2Q_2=\frac{1-K_1/K}{1-K_1/K_2}, \tag{59}$$

$$S_{11}=\alpha_1\frac{1-Q_1}{1-K_1/K_2}, \tag{60}$$

$$S_{22}=\alpha_2\frac{1-Q_2}{1-K_2/K_1}, \tag{61}$$

$$S_{12}=-\alpha_2\frac{v_2}{v_1}\frac{1-Q_2}{1-K_2/K_1}, \tag{62}$$

$$S_{21}=-\alpha_1\frac{v_1}{v_2}\frac{1-Q_1}{1-K_1/K_2}, \tag{63}$$

where K is defined as the overall drained bulk modulus of the two-phase composite [i.e., $K\equiv(\dot{\overline{p}}_c/\boldsymbol{\nabla}\cdot\mathbf{v})_{\dot{\overline{p}}_{f1},\dot{\overline{p}}_{f2}}$]. These results for the confining-pressure constants—which, importantly, are independent of the fluid's bulk modulus—allow the a_{ij} to be expressed as

$$a_{11}=\frac{1}{K}, \tag{64}$$

$$a_{22}=\frac{v_1\alpha_1}{K_1}\left(\frac{1}{B_1}-\frac{\alpha_1(1-Q_1)}{1-K_1/K_2}\right), \tag{65}$$

$$a_{33}=\frac{v_2\alpha_2}{K_2}\left(\frac{1}{B_2}-\frac{\alpha_2(1-Q_2)}{1-K_2/K_1}\right), \tag{66}$$

$$a_{12}=-\frac{v_1Q_1}{K_1}\alpha_1, \tag{67}$$

$$a_{13}=-\frac{v_2Q_2}{K_2}\alpha_2, \tag{68}$$

$$a_{23}=-\frac{\alpha_1\alpha_2K_1/K_2}{(1-K_1/K_2)^2}\left(\frac{1}{K}-\frac{v_1}{K_1}-\frac{v_2}{K_2}\right). \tag{69}$$

All dependence of these constants on the geometry of each phase and on the underlying shear moduli G_i is implicitly contained in K. All dependence on the fluid's bulk modulus is contained within the two Skempton's coefficients B_1 and B_2 and is thus restricted to a_{22} and a_{33}. The various experiments needed to measure the a_{ij} had been the focus of an earlier publication [21].

We emphasize that in the long-time limit where $\dot{\zeta}_{\text{int}}=0$ and, therefore, $\overline{p}_{f1}=\overline{p}_{f2}=\overline{p}_f$, the above double-porosity laws reduce to the single-porosity laws ($\boldsymbol{\nabla}\cdot\mathbf{q}=\boldsymbol{\nabla}\cdot\mathbf{q}_1+\boldsymbol{\nabla}\cdot\mathbf{q}_2$):

$$\begin{bmatrix}\boldsymbol{\nabla}\cdot\mathbf{v}\\ \boldsymbol{\nabla}\cdot\mathbf{q}\end{bmatrix}=-\begin{bmatrix}a_{11} & a_{12}+a_{13}\\ a_{12}+a_{13} & a_{22}+2a_{23}+a_{33}\end{bmatrix}\begin{bmatrix}\dot{\overline{p}}_c\\ \dot{\overline{p}}_f\end{bmatrix}. \tag{70}$$

These single-porosity results with the a_{ij} as established above are identical to the exact results of Berryman and Milton [43,44], who used different arguments.

VI. THE INTERNAL WAVE-INDUCED FLOW

From an attenuation perspective, the most important law is the one controlling the internal fluid transfer $\dot{\zeta}_{\text{int}}$ [Eq. (42)]. In the frequency domain, this law may be written as

$$-i\omega\zeta_{\text{int}}(\omega)=\gamma(\omega)[\overline{p}_{f1}(\omega)-\overline{p}_{f2}(\omega)]. \tag{71}$$

The frequency dependence of the transport coefficient $\gamma(\omega)$ is shown in Paper II to be

$$\gamma(\omega)=\gamma_o\sqrt{1-i\frac{\omega}{\omega_c}}, \tag{72}$$

where the detailed nature of the relaxation frequency ω_c and low-frequency limit γ_o are derived in Paper II.

VII. THE SHEAR RESPONSE

Upon applying a shear to an averaging element, it is possible to create local changes of the confining pressure and, therefore, fluid pressure (see, for example, Ref. [45]). An illustration is given in Fig. 2, in which a pure shear stress is applied to an averaging element containing an isolated ellipsoidal inclusion of phase 2. Although there is no confining-pressure change within the inclusion, there will be confining-pressure changes to the surrounding matrix phase 1. These lobes of enhanced and decreased fluid pressure will then equilibrate. General models for such shear-induced mesoscopic fluid flow apparently do not exist in the literature.

Unlike the compressional problem, not all mesoscopic geometries will result in shear-induced pressure gradients and local flow. For example, if the drained shear modulus is the same in both phases or for certain concentric geometries or models involving thin planar joints, no such local fluid-pressure gradients will be created (except possibly in the neighborhood of where the joints intersect, which is assumed to occupy negligible volume). In general, the shear-induced flow problem is more difficult than the compressional problem treated in Paper II. Our approach here is simply to acknowledge that such local flow is possible, but not to model it. Such an involved analysis must be left to a future study.

It is important to recognize that at least for isotropic composites, the applied shear will not result in any net change in the fluid pressure throughout either phase. As such, the mac-

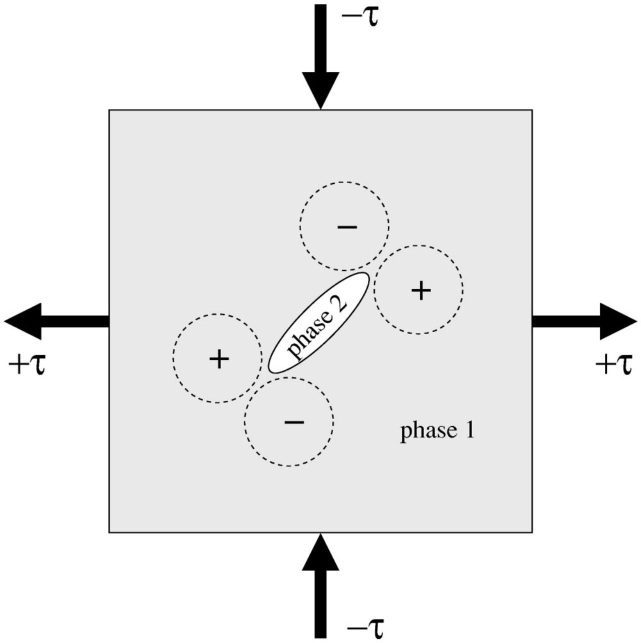

FIG. 2. A simple cartoon in which an ellipsoidal inclusion has a far-field shear stress suddenly applied to it resulting in the characteristic four-lobed zones of local fluid-pressure change that will subsequently equilibrate.

roscopic shear law takes the form (expressed for convenience in the frequency domain)

$$-i\omega\overline{\boldsymbol{\tau}}^{D}=[G(\omega)-i\omega F(\omega)]\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^{T}-\frac{2}{3}\nabla\cdot\mathbf{v}\,\mathbf{I}\right], \tag{73}$$

with $G(\omega)$ and $\omega F(\omega)$ both real functions that are Hilbert transforms of each other. The low-frequency limit of $G(\omega)$ corresponds to the drained-shear modulus of the composite while at high frequencies a larger unrelaxed shear modulus holds.

VIII. SUMMARY

The complete set of macroscopic equations governing the linear response of isotropic double-porosity composites with an assumed $e^{-i\omega t}$ time dependence is given by

$$\nabla\cdot\overline{\boldsymbol{\tau}}^{D}-\nabla\overline{p}_{c}=-i\omega(\rho\mathbf{v}+\rho_{f}\mathbf{q}_{1}+\rho_{f}\mathbf{q}_{2})-\rho\mathbf{g}, \tag{74}$$

$$\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\end{bmatrix}=-\frac{1}{\eta}\begin{bmatrix}\kappa_{11} & \kappa_{12}\\ \kappa_{12} & \kappa_{22}\end{bmatrix}\begin{bmatrix}\nabla\overline{p}_{f1}-\rho_f(i\omega\mathbf{v}+\mathbf{g})\\ \nabla\overline{p}_{f2}-\rho_f(i\omega\mathbf{v}+\mathbf{g})\end{bmatrix}, \tag{75}$$

$$\frac{1}{i\omega}\begin{bmatrix}\nabla\cdot\mathbf{v}\\ \nabla\cdot\mathbf{q}_1\\ \nabla\cdot\mathbf{q}_2\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{12} & a_{22} & a_{23}\\ a_{13} & a_{23} & a_{33}\end{bmatrix}\begin{bmatrix}\overline{p}_c\\ \overline{p}_{f1}\\ \overline{p}_{f2}\end{bmatrix}+\begin{bmatrix}0\\ \zeta_{\rm int}\\ -\zeta_{\rm int}\end{bmatrix}, \tag{76}$$

$$\zeta_{\rm int}=-\frac{\gamma(\omega)}{i\omega}(\overline{p}_{f1}-\overline{p}_{f2}), \tag{77}$$

$$-i\omega\overline{\boldsymbol{\tau}}^{D}=[G-i\omega F]\left[\nabla\mathbf{v}+(\nabla\mathbf{v})^{T}-\frac{2}{3}\nabla\cdot\mathbf{v}\,\mathbf{I}\right], \tag{78}$$

with $\gamma(\omega)=\gamma_o\sqrt{1-i\omega/\omega_c}$. All the coefficients appearing in these laws have been given definition earlier in the paper with the exception of the real shear coefficients $G(\omega)$ and $F(\omega)$. For wave problems, the acceleration of gravity can either be dropped from the description or used as a proxy for the seismic source. Berryman and Wang [3] have analyzed the wave properties of equations having this general form [though with $\dot{\zeta}_{\rm int}=0$ in Eq. (76)] and have demonstrated the existence of a second slow wave associated with the response in phase 2.

IX. AN UP-SCALED BIOT THEORY

One important use of this double-porosity theory is for constructing an effective, up-scaled, single-porosity theory that allows for the mesoscopic flow loss but that by its nature does not require analysis of a second slow wave in the second porous phase.

A simple way to construct such an up-scaled "Biot theory" is to assume that phase 2 is entirely embedded within phase 1 so that it does not intersect the external surface of an averaging volume. In this case, $\nabla\cdot\mathbf{q}_2=0$ and it is straightforward to eliminate $\overline{p}_{f2}$ from the above equations. The macroscopic response of the composite is then controlled by equations having exactly the form of Eqs. (1)–(4) but with an effective complex drained modulus $K_D(\omega)$, undrained modulus $K_U(\omega)$, and Skempton's coefficient $B(\omega)$ given by

$$\frac{1}{K_D}=a_{11}-\frac{a_{13}^2}{a_{33}-\gamma/i\omega}, \tag{79}$$

$$B=\frac{-a_{12}(a_{33}-\gamma/i\omega)+a_{13}(a_{23}+\gamma/i\omega)}{(a_{22}-\gamma/i\omega)(a_{33}-\gamma/i\omega)-(a_{23}+\gamma/i\omega)^2}, \tag{80}$$

$$\frac{1}{K_U}=\frac{1}{K_D}+B\left(a_{12}-\frac{a_{13}(a_{23}+\gamma/i\omega)}{a_{33}-\gamma/i\omega}\right). \tag{81}$$

One then obtains the effective Biot-Willis constant using the standard definition $\alpha=(1-K_D/K_U)/B$. The effective fluid permeability is defined by applying a single fluid-pressure drop across a sample of the two-phase composite. Various effective-medium approximations could be invoked; however, a simple harmonic mean $1/k=v_1/k_1+v_2/k_2$ is appropriate for the situation we are considering, where phase 2 is being modeled as embedded entirely within phase 1.

X. EXAMPLES OF ACOUSTIC ATTENUATION

Some examples of P-wave attenuation in a double-porosity composite are now given in which phase 2 is modeled as a small sphere of radius a_2 embedded at the center of

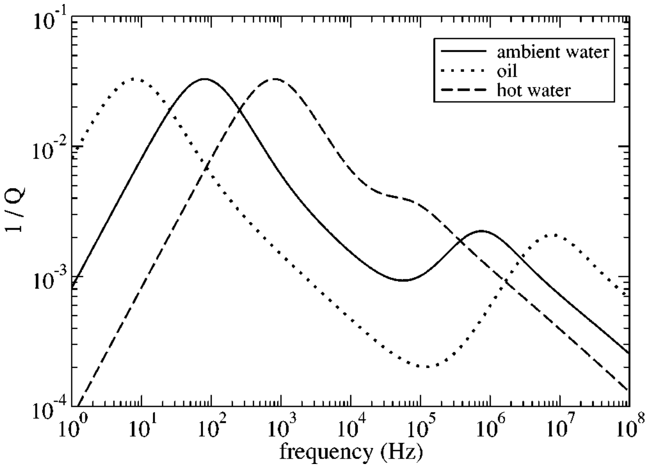

FIG. 3. The *P*-wave attenuation of a double-porosity composite having properties discussed in the text. The three curves correspond to viscosities of $\eta = 10^{-3}$ Pa s (solid curve representing ambient water), 2×10^{-4} Pa s (dashed curve representing hot water), and 5×10^{-3} Pa s (dotted curve representing oil).

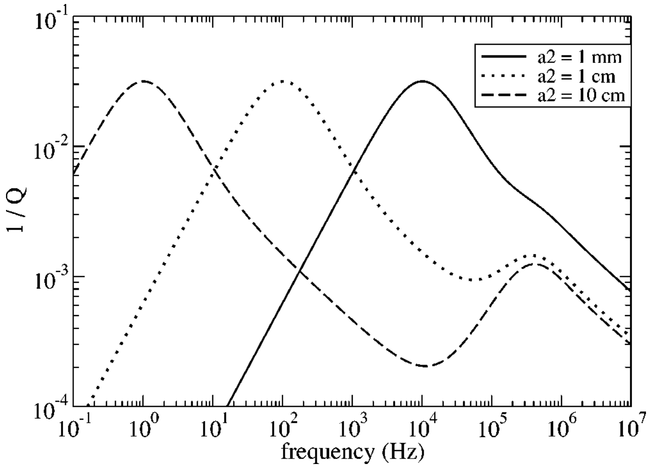

FIG. 4. The *P*-wave attenuation of a double-porosity composite when the radius a_2 of the spherical inclusion of phase 2 varies at fixed ratio $R = 3a_2$ (i.e., at constant volume fraction $v_2 = 3.7\%$). The three curves correspond to three different values of a_2 as shown. It is not possible to consider smaller spheres than 1 mm because the notion of a porous continuum requires at least a few tens of grains to be present in each phase.

each sphere of composite having radius *R*. We fix $R = 3a_2$ in all examples so that $v_2 = (a_2/R)^3 \approx 0.037$ and $v_1 = 0.963$. The volume to internal surface ratio V/S present in the relaxation frequency ω_c as defined in Paper II is given by $V/S = R^3/(3a_2^2) = 9\, a_2$.

We take the composite's drained modulus $1/a_{11}$ to be a harmonic average of the two constituent drained moduli that are in turn determined from the effective-medium formula $K_i = (1 - \phi_i)K_s/(1 + c_i\phi_i)$ where the parameters c_i may be called "consolidation" parameters. We take the low-frequency fluid-pressure gradient length in phase 1 [that is present in the expression for γ_o and is defined in Eq. (37) of Paper II] to be $L_1 = a_2\sqrt{99/28} \approx 1.9a_2$ when $R = 3a_2$.

The embedded phase 2 is assumed to represent small pockets where the grains are less-well consolidated. Accordingly, we model it with numbers corresponding to a poorly consolidated sandstone $\phi_2 = 0.30$, $k_2 = 10^{-12}$ m^2, and $c_2 = 200$. For the matrix phase 1, we take the values appropriate to a consolidated shaly sandstone $\phi_1 = 0.10$, $k_1 = 10^{-14}$ m^2, and $c_1 = 10$. In the following examples, we assume that the composite's shear modulus is given by $G = (1 - \phi)G_s/(1 + c_d\phi)$, where $\phi = v_1\phi_1 + v_2\phi_2$ and where $c_d = 10$. We take the permeability to be given by $1/k = v_1/k_1$.

In the first example given in Fig. 3, we fix $a_2 = 1$ cm and vary the fluid viscosity to show how the two peaks in the attenuation Q^{-1} vary (note that Q^{-1} represents the fraction of energy irreversibly lost to heat in each wave period). The peak to the left for each curve corresponds to the frequency when the mesoscopic structure just has time to equilibrate in one cycle while the peak to the right corresponds to the Biot-loss maximum, which occurs when the entire wavelength of fluid-pressure variation just equilibrates in a cycle. Note that the effect of viscosity is to shift the peaks differently. Since the mesoscopic heterogeneity has some characteristic length $L_{\text{meso}} \approx a_2$, it will equilibrate when the frequency is at $f_{\text{meso}} = D/L_{\text{meso}}^2$ and since the pore-pressure diffusivity *D* is inversely proportional to viscosity, the mesoscopic peaks shift downward with increasing viscosity. For the Biot loss, the distance L_{Biot} over which the pressure equilibrates is the wavelength λ, so that $L_{\text{Biot}} = v_p/f$ where v_p is the *P*-wave velocity and thus, equilibration will just occur when $f_{\text{Biot}} = Df_{\text{Biot}}^2/v_p^2$ or in otherwords when $f_{\text{Biot}} = v_p^2/D$, which has just the opposite dependence on the fluid viscosity and permeability than does the mesoscopic flow.

In the second example given in Fig. 4, we keep the porous-continuum properties the same (with ambient water in the pores) but change the size a_2 of the embedded sphere while keeping the volume fractions the same ($R = 3a_2$).

XI. CONCLUSIONS

To conclude this part of our analysis, we emphasize that the so-called "Biot loss" is the pressure equilibration occurring between the peaks and troughs of a compressional wave when the porous material is taken to be uniform over the wavelength. When a compressional wave squeezes an element of material containing mesoscopic heterogeneity, there is induced a heterogeneous fluid-pressure response within the element that then equilibrates. As has been demonstrated here using a double-porosity model of such mesoscopic heterogeneity, considerable low-frequency attenuation can be generated by this equilibration. The peak value of Q^{-1} in the double-porosity mechanism is controlled principally by the contrast of the drained bulk modulus between the two phases and is independent of the permeability of the materials. However, the relaxation frequency at which Q^{-1} peaks is directly proportional to the permeability k_1 of the host phase 1. Furthermore, at frequencies less than the relaxation frequency, Q^{-1} increases in proportion to $\omega\eta/k_1$. This is exactly inverse to the dependence on η/k found in the Biot wavelength-scale mechanism.

Three natural extensions of the present two-porosity theory include allowing for anisotropic composites, generalizing to *N*-porosity composites, and introducing the proper fluid-dependent shear response.

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